The potential, electric field and surface charges for a resistive long straight strip carrying a steady current

J. A. Hernandes[a)]
Instituto de Física "Gleb Wataghin," Universidade Estadual de Campinas-Unicamp, 13083-970 Campinas, São Paulo, Brazil

A. K. T. Assis[b)]
Institut für Geschichte der Naturwissenschaften, Universität Hamburg, Bundesstrasse 55, D-20146 Hamburg, Germany

(Received 20 March 2002; accepted 24 March 2003)

We consider a long resistive straight strip carrying a constant current and calculate the potential and electric field everywhere in space and the density of surface charges along the strip. We compare these calculations with experimental results.
[DOI: 10.1119/1.1574319]

I. THE PROBLEM

Recently there has been renewed interest in the electric field outside stationary resistive conductors carrying a constant current.[1–7] We consider a case that has not been treated in the literature, namely, a constant current flowing uniformly over the surface of a stationary and resistive straight strip. Our goal is to calculate the potential ϕ and electric field $\mathbf{E}$ everywhere in space and the surface charge distribution σ along the strip that creates this electric field.

We consider a strip in the $y=0$ plane localized in the region $-a<x<a$ and $-\ell<z<\ell$, such that $\ell \gg a>0$. The medium around the strip is taken to be air or vacuum. The constant current I flows uniformly along the positive z direction with a surface current density given by $\mathbf{K}=I\hat{z}/2a$ (see Fig. 1). By Ohm's law this uniform current distribution is related to a spatially constant electric field on the surface of the strip. In the steady state this electric field can be related to the potential by $\mathbf{E}=-\nabla\phi$. This relation means that along the strip the potential is a linear function of z and independent of x. The problem can then be solved by finding the solution of Laplace's equation $\nabla^2\phi=0$ in empty space and applying the boundary conditions.

II. THE SOLUTION

Due to the symmetry of the problem, it is convenient to use elliptic-cylindrical coordinates (η, φ, z).[8] These variables can take the following values: $0\leqslant\eta\leqslant\infty$, $0\leqslant\varphi\leqslant 2\pi$, and $-\infty\leqslant z\leqslant\infty$. The relation between Cartesian (x, y, z) and elliptic-cylindrical coordinates is given by

$$x=a\cosh\eta\cos\varphi, \tag{1a}$$

$$y=a\sinh\eta\sin\varphi, \tag{1b}$$

$$z=z, \tag{1c}$$

where a is the constant semi-thickness of the strip. The inverse relations are given by

$$\eta=\tanh^{-1}\sqrt{\frac{x^2-y^2-a^2+\Omega}{2x^2}}, \tag{2a}$$

$$\varphi=\tan^{-1}\sqrt{\frac{a^2-x^2+y^2+\Omega}{2x^2}}, \tag{2b}$$

$$z=z, \tag{2c}$$

where $\Omega=\sqrt{(x^2+y^2+a^2)^2-4a^2x^2}$.

Laplace's equation in this coordinate system is given by

$$\nabla^2\phi=\frac{1}{a^2(\cosh^2\eta-\cos^2\varphi)}\left(\frac{\partial^2\phi}{\partial\eta^2}+\frac{\partial^2\phi}{\partial\varphi^2}\right)+\frac{\partial^2\phi}{\partial z^2}=0. \tag{3}$$

A solution of Eq. (3) can be obtained by using separation of variables in the form $\phi(\eta,\varphi,z)=H(\eta)\Phi(\varphi)Z(z)$:

$$H''-(\alpha_2+\alpha_3a^2\cosh^2\eta)H=0, \tag{4a}$$

$$\Phi''+(\alpha_2+\alpha_3a^2\cos^2\varphi)\Phi=0, \tag{4b}$$

$$Z''+\alpha_3Z=0, \tag{4c}$$

where α_2 and α_3 are constants.

For a long strip being considered here, it is possible to neglect boundary effects near $z=\pm\ell$. It has already been proved that in this case the potential must be a linear function of z, not only over the strip, but also over all space.[9] This condition means that $\alpha_3=0$. There are then two possible solutions for $\Phi(\varphi)$. If $\alpha_2=0$, then $\Phi=C_1+C_2\varphi$; if $\alpha_2\neq 0$, then $\Phi=C_3\sin\sqrt{\alpha_2}\varphi+C_4\cos\sqrt{\alpha_2}\varphi$, where C_1 to C_4 are constants. Along the strip we have $y=0$, and $x^2\leqslant a^2$, which means that $\Omega=a^2-x^2$, $\eta=0$, and $\varphi=\tan^{-1}\sqrt{(a^2-x^2)/x^2}$. Because the potential does not depend on x along the strip, this independence means that the potential will not depend on φ as well. Thus a nontrivial solution for Φ can only exist if $\alpha_2=0$, $C_2=0$, and $\Phi=$constant for all φ. The solution for H with $\alpha_2=\alpha_3=0$ will be then a linear function of η. The general solution of the problem is then given by

$$\phi=(A_1\eta-A_2)(A_3z-A_4)$$
$$=\left[A_1\tanh^{-1}\sqrt{\frac{x^2-y^2-a^2+\Omega}{2x^2}}-A_2\right](A_3z-A_4). \tag{5}$$

The electric field $\mathbf{E}=-\nabla\phi$ takes the following form:

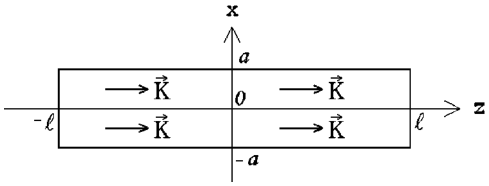

Fig. 1. A constant current I flows along the z direction of a long straight strip of length 2ℓ and width $2a$ located at $y=0$, with a surface current density given by $\mathbf{K}=I\hat{z}/2a$.

$$\mathbf{E}=-A_1\left(\frac{|x|\sqrt{x^2-y^2-a^2+\Omega}}{x\sqrt{2}\Omega}\hat{x}+\frac{|x|y\sqrt{2}}{\Omega\sqrt{x^2-y^2-a^2+\Omega}}\hat{y}\right)(A_3z-A_4)-A_3\left(A_1\tanh^{-1}\sqrt{\frac{x^2-y^2-a^2+\Omega}{2x^2}}-A_2\right)\hat{z}. \tag{6}$$

To find the surface charge density, we utilized the approximation close to the strip ($|x|<a$ and $|y|\ll a$):

$$\mathbf{E}\approx -A_1\left[\frac{x|y|}{(a^2-x^2)^{3/2}}\hat{x}+\frac{y}{|y|\sqrt{a^2-x^2}}\hat{y}\right](A_3z-A_4)-A_3\left[A_1\tanh^{-1}\frac{|y|}{\sqrt{a^2-x^2}}-A_2\right]\hat{z}. \tag{7}$$

The surface charge density $\sigma(x,z)$ can be obtained by the standard procedure utilizing Gauss's law $\oint_S \mathbf{E}\cdot d\mathbf{a}=Q/\varepsilon_0$, where ε_0 is the vacuum permittivity, $d\mathbf{a}$ is a surface area element pointing outward normal to the surface in each point, and Q is the total charge inside the closed surface S. The surface charge density is then obtained by considering the limit in which $|y|\to 0$ in Eq. (7) and a small cylindrical volume with its length much smaller then its diameter, yielding: $\sigma=\epsilon_0[\mathbf{E}(y>0)\cdot\hat{y}-\mathbf{E}(y<0)\cdot(-\hat{y})]$. If we use Eq. (7), the surface charge density is found to be given by

$$\sigma=-\frac{2\epsilon_0 A_1(A_3z-A_4)}{\sqrt{a^2-x^2}}. \tag{8}$$

III. DISCUSSION

In the plane $y=0$ the current in the strip creates a magnetic field $\mathbf{B}$ that points along the positive (negative) y direction for $x>0$ ($x<0$). This magnetic field will act on the conduction electrons moving with drift velocity $\mathbf{v}_d$ with a force given by $q\mathbf{v}_d\times\mathbf{B}$ (see Fig. 2). This force will cause a redistribution of charges along the x direction, with negative charges concentrating along the center of the strip and positive charges at the extremities $x=\pm a$. In the steady state this redistribution of charges will create an electric field along the x direction, E_x, that will balance the magnetic force, namely, $|qE_x|=|qv_dB|$.

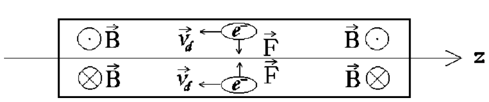

Fig. 2. Magnetic force $\mathbf{F}=q\mathbf{v}_d\times\mathbf{B}$ directed along the center of the strip acting on a conduction electron moving with drifting velocity $\mathbf{v}_d$. This force is due to the magnetic field $\mathbf{B}$ generated by the electric current I flowing along the positive z direction.

We have disregarded this Hall electric field because it is usually much smaller than the electric field giving rise to the current.[10] To estimate the orders of magnitude involved, it is easier to consider the current I flowing uniformly in a long cylinder of length 2ℓ and radius a along the positive z direction coinciding with the axis of this cylinder. This current generates a cylindrical magnetic field given by (at a distance $r<a$ from the axis) $\mathbf{B}=\mu_0 Ir\hat{\varphi}/2\pi a^2$, where μ_0 is the vacuum permeability and $\hat{\varphi}$ is the unit polar vector. The magnetic force acting on an electron of charge $q=-e$ moving with drift velocity $\mathbf{v}_d=-|\mathbf{v}_d|\hat{z}$ relative to the lattice of the wire is given by $q\mathbf{v}_d\times\mathbf{B}=-|\mu_0 ev_dIr/2\pi a^2|\hat{r}$, where $\hat{r}$ is the unit radial vector. This inward radial force will lead to an accumulation of negative charges in the interior of the wire, which creates a radial electric field $\mathbf{E}_r$ pointing inward. In the steady state the electric and magnetic radial forces will balance one another, $q\mathbf{E}_r=q\mathbf{v}_d\times\mathbf{B}$, yielding $\mathbf{E}_r=-|\mu_0 v_dIr/2\pi a^2|\hat{r}$. This electric field increases linearly inside the wire. Its maximum value close to $r=a$ is given by $|E_r^{\max}|=|\mu_0 v_dI/2\pi a|$. The longitudinal electric field giving rise to the current can be obtained by Ohm's law, $V=RI$, where V is the electromotive force along the wire of resistance R. For a wire of length 2ℓ acted on by a longitudinal electric field $|E_\ell|$ pointing along the z direction, this voltage is given by $V=2\ell|E_\ell|$, such that $|E_\ell|=RI/2\ell$. The ratio between the maximal radial electric field and the longitudinal one is given by $|E_r^{\max}/E_\ell|=|\mu_0 v_d\ell/\pi aR=\mu_0 v_d ga/2|$, where g is the conductivity of the wire and is related to its resistance by $R=2\ell/g\pi a^2$. We use the notation g instead of the more standard notation σ for the conductivity in order to avoid confusion with the surface charge density, which is represented by σ.

To find the order of magnitude, we consider a copper wire ($g=5.7\times10^7\ \Omega$ m and $v_d\approx 4\times10^{-3}\ \mathrm{m\,s^{-1}}$) of 1 mm diameter ($a=5\times10^{-4}$ m). With these values in Eq. (8), we obtain $|E_r^{\max}/E_\ell|\approx 7\times10^{-5}$, justifying our neglect of the radial component of the electric field. Conceptually this neglect of the radial component can be explained by the fact that the Hall electric field is small because it is due to the small magnetic field produced by the conducting strip, rather than a large applied magnetic field.

We now analyze some particular cases. We first consider two limits by comparing a with the distance of the observation point $r=\sqrt{x^2+y^2}$. If $a^2\gg r^2$, we have $\Omega\approx a^2+y^2-x^2+2x^2y^2/a^2$ and $\eta\approx|y|/a$, such that

$$\phi\approx\left(A_1\frac{|y|}{a}-A_2\right)(A_3z-A_4). \tag{9}$$

As expected, this result coincides with Eq. (4) of Ref. 11 with $y_0=0$, because only the case $a^2\gg r^2$ was considered there.

On the other hand, if $a^2\ll r^2$, we have $\Omega\approx r^2+a^2-2a^2x^2/r^2$ and $\eta\approx\ln r/a$, such that

$$\phi\approx\left(A_1\ln\frac{r}{a}-A_2\right)(A_3z-A_4). \tag{10}$$

This result coincides with Eq. (8) of Ref. 10 with $A_2/A_1 = \ln(2\ell/a)$, where ℓ is the typical length of the wire or strip being considered, with $\ell \gg a$. [Note that in Ref. 10 the length of the wire along z goes from $-\ell/2$ to $\ell/2$, while here it goes from $-\ell$ to ℓ (see Fig. 1).] This coincidence is reasonable because Eq. (8) of Ref. 10 corresponds to the potential outside a long straight cylindrical wire carrying a constant current. At a point far from the axis of the strip, both results coincide as they must.

Equation (6) indicates that there is an electric field not only along the resistive strip carrying a constant current, but also in the space surrounding it. Jefimenko has peformed some experiments that show the existence of this external electric field. The geometry of his first experiment,[12] reproduced in plate 6 of Ref. 13, is equivalent to what has been considered here: a two-dimensional conducting strip made on a glass plate using a transparent conducting ink. To compare our calculations with his experimental results, we need the values of A_2/A_1 and A_4/A_3. We take $A_2/A_1=3.6$ and $A_4/A_3=0$. The condition $A_4/A_3=0$ corresponds to the symmetrical case considered by Jefimenko in which the electric field is parallel to the conductor just outside of it at $z=0$ (zero density of surface charges at $z=0$).

We first consider the plane orthogonal to the strip, $x=0$. In this case the potential reduces to

$$\phi=\left(A_1 \tanh^{-1}\sqrt{\frac{y^2}{y^2+a^2}}-A_2\right)(A_3 z-A_4). \tag{11}$$

The lines of the electric field orthogonal to the equipotentials can be obtained by the procedure in Ref. 14. These lines are represented by a function ψ such that $\nabla\psi\cdot\nabla\phi=0$. Equation (11), together with the value of ϕ obtained above, yield the value of ψ given by

$$\begin{aligned}\psi={}&A_1A_3z^2-2A_1A_4z+\frac{A_1A_3}{2}y^2\\&-A_1A_3|y|\sqrt{y^2+a^2}\cosh^{-1}\sqrt{\frac{y^2+a^2}{a^2}}\\&-\frac{A_1A_3}{2}a^2\left(\cosh^{-1}\sqrt{\frac{y^2+a^2}{a^2}}\right)^2\\&-\frac{A_2A_3}{4}\left(|y|\sqrt{y^2+a^2}+a^2\ln\frac{|y|+\sqrt{y^2+a^2}}{a}\right).\end{aligned} \tag{12}$$

A plot of Eqs. (11) and (12) is given in Fig. 3.

We now consider the plane of the strip, $y=0$. The potential reduces to

$$\phi(|x|\leqslant a,0,z)=-A_2(A_3z-A_4), \tag{13}$$

$$\begin{aligned}\phi(|x|\geqslant a,0,z)&=\left(A_1\tanh^{-1}\sqrt{\frac{x^2-a^2}{x^2}}-A_2\right)(A_3z-A_4)\\&=\left(A_1\cosh^{-1}\frac{|x|}{a}-A_2\right)(A_3z-A_4).\end{aligned} \tag{14}$$

When there is no current in the strip, the potential along it is a constant for all z. From Eq. (13) this condition implies that $A_3=0$. This value of A_3 in Eqs. (5), (6), and (8) reduces these equations to the known electrostatic solution of a strip charged to a constant potential.[15]

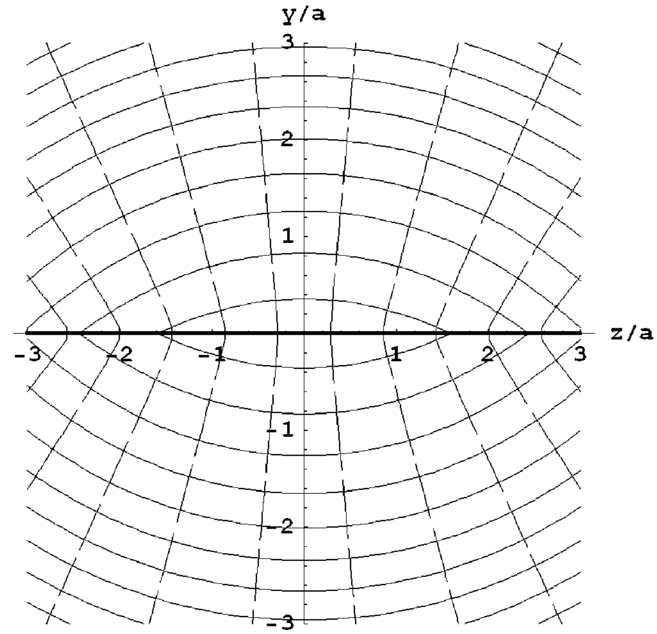

Fig. 3. Equipotential lines (dashed) and electric field lines (continuous) in the $x=0$ plane. The bold horizontal lines represent the intersection with the plane of the strip. We use the values $A_2/A_1=3.6$ and $A_4/A_3=0$.

By a similar procedure, the lines of electric field for the plane $y=0$ are given by

$$\psi(|x|\leqslant a,0,z)=A_2A_3ax, \tag{15}$$

$$\begin{aligned}\psi(|x|\geqslant a,0,z)={}&A_1A_3z^2-2A_1A_4z+\frac{A_1A_3}{2}x^2\\&-A_1A_3|x|\sqrt{x^2-a^2}\cosh^{-1}\frac{|x|}{a}\\&+\frac{A_1A_3}{2}a^2\left(\cosh^{-1}\frac{|x|}{a}\right)\\&-\frac{A_2A_3}{4}\left(|x|\sqrt{x^2-a^2}\right.\\&\left.-a^2\ln\frac{|x|+\sqrt{x^2-a^2}}{a}\right).\end{aligned} \tag{16}$$

A plot of Eqs. (13)–(16) is presented in Fig. 4. Figure 5 presents the theoretical electric field lines and equipotential lines overlaid on the experimental results of Ref. 12, where the lines of the electric field in the plane of the strip are mapped by spreading grass seeds above and around the two-dimensional conducting strip painted on glass plates. The seeds are polarized in the presence of an electric field and align themselves with it. The lines of electric field are then observed in analogy with iron fillings generating the lines of magnetic field. In Fig. 5 the electric field lines from Fig. 4 are overlaid on the experimental results of Jefimenko, Fig. 1 of Ref. 12 or plate 6 of Ref. 13. It should be mentioned that the grass seeds are dielectric bodies and themselves change the electric fields in their vicinity, so the experimental field maps cannot be exact; nevertheless, the correspondence found here is reasonable.

The equipotential lines also were measured in Ref. 16 where a rectangular hollow chamber with electrodes (alumi-

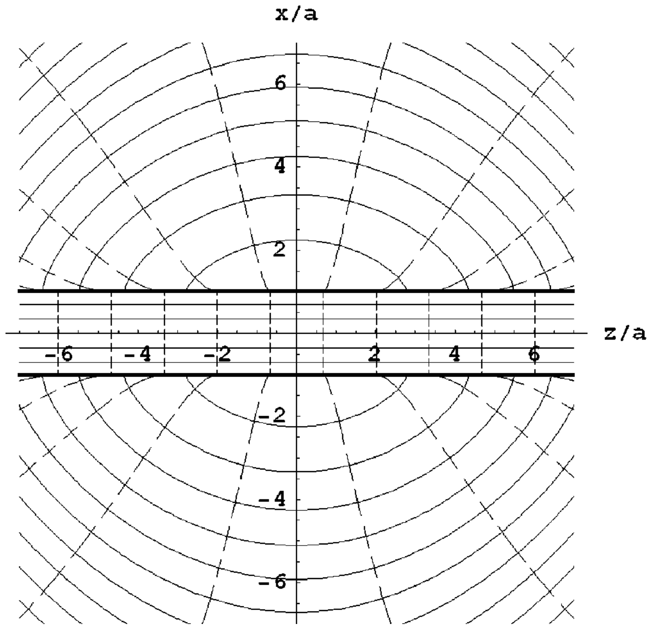

Fig. 4. Equipotential lines (dashed) and electric field lines (continuous) in the $y=0$ plane. The bold horizontal lines represent the boundaries of the strip at $x/a=1$ and $x/a=-1$. We assume $A_2/A_1=3.6$ and $A_4/A_3=0$.

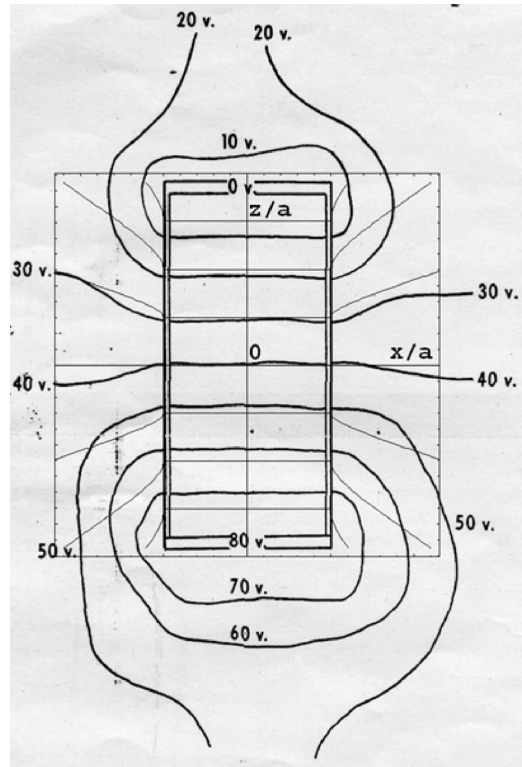

Fig. 6. Equipotential lines in the $y=0$ plane overlaid on Fig. 3(a) of Ref. 16. We use $A_2/A_1=3.0$ and $A_4/A_3=0$.

num foil) for end walls and semi-conducting side walls (graphite paper strips) carrying uniform current was used. Eighty volts were applied to the electrodes and the equipotential lines were mapped utilizing a radioactive alpha source to ionize the air at the points where the field was to be measured. The alpha source acquired the same potential as the field at those points and the potential was measured with an electronic electrometer connected to the alpha source. In Fig. 6 the experimental result of Ref. 16 is superimposed on the equipotential lines calculated utilizing Eqs. (15) and (16) with $A_2/A_1=3.0$ and $A_4/A_3=0$. The agreement is not as good as in Fig. 5 for two reasons: One reason is that our calculations are for a two-dimensional geometry, while the experiment in Ref. 16 was performed in a three-dimensional rectangular chamber. The second reason is that in the grass seed experiment[12] the ratio of the length to the thickness of the conductor was 7, but in the second experiment[16] this ratio

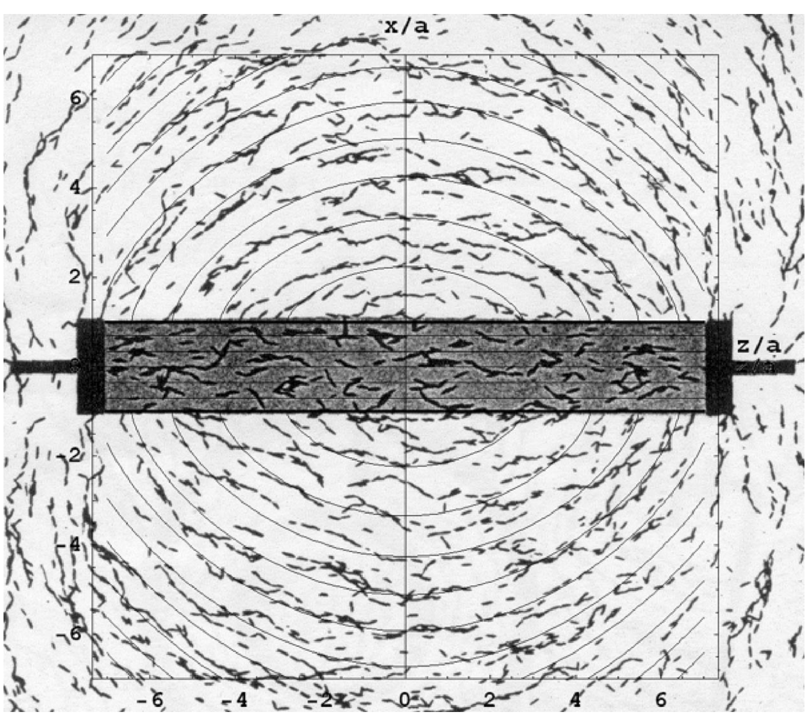

Fig. 5. Electric field lines of Fig. 4 overlaid on plate 6 of Ref. 13.

was only 2.3, which means that boundary effects near $z=\ell$ and $z=-\ell$ are more important. These boundary effects were not considered in our calculations.

One of the main aspects of this work is that we succeeded in finding a theoretical model yielding reasonable results which were compared with two different experiments already published in the literature. As discussed above, these experiments mapped the electric field lines and the equipotential lines inside and outside the regions of steady currents in conductors. The geometry considered here had never been dealt with in this problem before. In order to obtain this result it was necessary to use elliptic-cylindrical coordinates (η, φ, z). The general solution for the potential in terms of these variables is reasonably simple, namely, $\phi=(A_1\eta-A_2)(A_3z-A_4)$. When expressed in terms of the usual Cartesian coordinates (x, y, z) the solution takes the complicated form of Eq. (5). We could not obtain this solution working only with cartesian coordinates. In this problem the pure cylindrical coordinates are not so practical as well. The situation described here shows an important example of the usefulness of the elliptic-cylindrical coordinates in dealing with reasonably simple problems of physics.

ACKNOWLEDGMENTS

The authors thank the referees for important suggestions. JAH thanks CNPq (Brasil) for financial support. AKTA thanks the Alexander von Humboldt Foundation of Germany for a research fellowship during which this work was accomplished.

[a]Electronic mail: julioher@ifi.unicamp.br
[b]Permanent address: Instituto de Física "Gleb Wataghin," Universidade Estadual de Campinas Unicamp, 13083-970 Campinas, São Paulo, Brazil; electronic mail: assis@ifi.unicamp.br

[1]M. A. Heald, "Electric fields and charges in elementary circuits," Am. J. Phys. **52**, 522–526 (1984).
[2]J. M. Aguirregabiria, A. Hernandez, and M. Rivas, "An example of surface-charge distribution on conductors carrying steady currents," Am. J. Phys. **60**, 138–141 (1992).
[3]A. K. Singal, "The charge neutrality of a conductor carrying a steady current," Phys. Lett. A **175**, 261–264 (1993).
[4]J. D. Jackson, "Surface charges on circuit wires and resistors play three roles," Am. J. Phys. **64**, 855–870 (1996).
[5]A. K. T. Assis and A. J. Mania, "Surface charges and electric field in a two-wire resistive transmission line," Rev. Bras. Ensino de Física **21**, 469–475 (1999).
[6]A. K. T. Assis and J. I. Cisneros, "Surface charges and fields in a resistive coaxial cable carrying a constant current," IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **47**, 63–66 (2000).
[7]N. W. Preyer, "Surface charges and fields of simple circuits," Am. J. Phys. **68**, 1002–1006 (2000).
[8]P. Moon and D. E. Spencer, *Field Theory Handbook* (Springer, Berlin, 1988), 2nd ed., p. 17.
[9]B. R. Russell, "Surface charges on conductors carrying steady currents," Am. J. Phys. **36**, 527–529 (1968).
[10]A. K. T. Assis, W. A. Rodrigues, Jr., and A. J. Mania, "The electric field outside a stationary resistive wire carrying a constant current," Found. Phys. **29**, 729–753 (1999).
[11]A. K. T. Assis, J. A. Hernandes, and J. E. Lamesa, "Surface charges in conductor plates carrying constant currents," Found. Phys. **31**, 1501–1511 (2001).
[12]O. D. Jefimenko, "Demonstration of the electric fields of current-carrying conductors," Am. J. Phys. **30**, 19–21 (1962).
[13]O. D. Jefimenko, *Electricity and Magnetism* (Electret Scientific, Star City, 1989), 2nd ed.
[14]A. Sommerfeld, *Electrodynamics* (Academic, New York, 1964), p. 128.
[15]D. J. Griffiths and Y. Li, "Charge density on a conducting needle," Am. J. Phys. **64**, 706–714 (1996).
[16]O. Jefimenko, T. L. Barnett, and W. H. Kelly, "Confinement and shaping of electric fields by current-carrying conductors," in Proceedings of the West Virginia Academy of Science, 1962, Vol. 34, pp. 163–167.