Electric potential for a toroidal ring carrying a constant current

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Abstract

In this work we treat a conducting toroidal ring carrying a constant current. We calculate the electric potential everywhere in space. We compare our theoretical result with Jefimenko's experiment.

1 Description of the problem

Consider a stationary toroidal ring conducting a constant current I, uniformly distributed over the cross section of the conductor. The ring is centered on $z = 0$, being z the symmetry axis of the ring. Our goal is to find the electric potential everywhere in space, using the potential in the surface of the conductor as a boundary condition.

The symmetry of our problem suggests the approach of toroidal coordinates (η, θ, φ) [1, page 112], defined by:

$$x = a\frac{\sinh\eta\cos\varphi}{\cosh\eta - \cos\theta}, \qquad y = a\frac{\sinh\eta\sin\varphi}{\cosh\eta - \cos\theta}, \qquad z = a\frac{\sin\theta}{\cosh\eta - \cos\theta}. \tag{1}$$

Laplace's equation for the electric potential outside the conductor ring, $\nabla^2\phi = 0$, can be solved in toroidal coordinates with separation of variables, leading to a solution of the form:

$$\phi(\eta, \theta, \varphi) = \sqrt{\cosh\eta - \cos\theta}\, H(\eta)\Theta(\theta)\Phi(\varphi), \tag{2}$$

where the functions H, Θ and Φ satisfy the general equations (with $\xi = \cosh\eta$, and p and q are constants):

$$\left\{ \begin{array}{rcl} (\xi^2 - 1)H'' + 2\xi H' - \left[(p^2 - \frac{1}{4}) + \frac{q^2}{\xi^2 - 1}\right] H & = & 0, \\ \Theta'' + p^2\Theta & = & 0, \\ \Phi'' + q^2\Phi & = & 0. \end{array} \right. \tag{3}$$

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2 Discussion of the Equations and Solution

The equation for the function H is Legendre's equation, so H is Legendre's function with semi-integer indices: $P^q_{p-1/2}$ and $Q^q_{p-1/2}$, known as toroidal Legendre polynomials, [2].

As the solutions for Legendre's equation $Q^q_{p-1/2}$ are irregular on $\eta = 0$ (that is, for the z axis and for great distances from the ring), we eliminate them as physical solutions for this problem. Our general solution consists of a linear combination of all possible regular solutions of H, Θ and Φ.

In particular, we consider the potential along the ring surface to be linear with φ. The ring surface is described by a constant η_0. Therefore, this potential can be expanded in Fourier series, [3]:

$$\phi(\eta_0, \theta, \varphi) = \phi_0 \frac{\varphi}{2\pi} = \frac{\phi_0}{\pi} \sum_{q=1}^{\infty} \frac{(-1)^q}{q} \sin(q\varphi). \tag{4}$$

We use this equation as a boundary condition to our problem, yielding as final solution:

$$\phi(\eta, \theta, \varphi) = \sqrt{\cosh\eta - \cos\theta} \sum_{q=1}^{\infty} \sin(q\varphi) \sum_{p=0}^{\infty} A_{pq} P^q_{p-1/2}(\cosh\eta)\cos(p\theta). \tag{5}$$

The coefficients A_{pq} are given by, for $p = 0$ and $p > 0$, respectively:

$$A_{0q} = \frac{\phi_0(-1)^{q-1}}{2\pi^2 q P^q_{-1/2}(\cosh\eta_0)} \int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{\cosh\eta_0 - \cos\theta}}, \tag{6}$$

$$A_{pq} = \frac{\phi_0(-1)^{q-1}}{\pi^2 q P^q_{p-1/2}(\cosh\eta_0)} \int_{-\pi}^{\pi} \frac{\cosh(p\theta)d\theta}{\sqrt{\cosh\eta_0 - \cos\theta}}. \tag{7}$$

We plotted the equipotentials on the plane of the ring in Figure 1. This Figure can then be compared with the experimental result found by Jefimenko, [4, Figure 3] reproduced here in Figure 2. There is a very reasonable agreement between our theoretical result and the experiment.

3 Thin Charged Wire without Current

We do the same calculations as above, but for a charged ring without current (constant potential on the ring surface). We then make an approximation for a thin ring, $\eta_0 \gg 1$, and we obtain:

$$\phi(\eta, \theta) = \phi_0 \sqrt{\frac{\cosh\eta - \cos\theta}{\cosh\eta_0}} \frac{P_{-1/2}(\cosh\eta)}{P_{-1/2}(\cosh\eta_0)}. \tag{8}$$

This expression can be compared with that given by Jackson, [5, p. 93], in spherical coordinates (r, θ, ϕ):

$$\phi(r, \theta) = q \sum_{\lambda=0}^{\infty} \frac{r_<^{2\lambda}}{r_>^{2\lambda+1}} \frac{(-1)^\lambda (2\lambda - 1)!!}{2^\lambda \lambda!} P_{2\lambda}(\cos\theta), \tag{9}$$

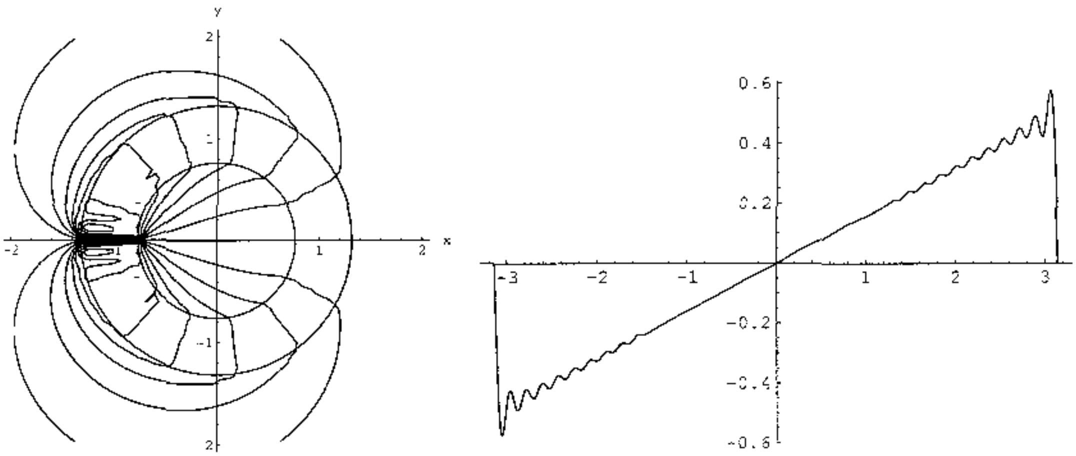

Figure 1: On the left are the equipotentials on the plane $z = 0$. On the right, the potential along the ring surface (should be a linear function of φ). Note the inevitable oscillations due to Fourier expansion [6, p. 836].

where $r_<$ ($r_>$) is the lesser (greater) between r and a, a is the radius of the ring, and q is the total charge of the ring. We plotted both Eq. (8) and (9) in Figure 3 and we obtained the same result for both. We should stress though that in spherical coordinates we have an infinite sum, Eq. (9), while in toroidal coordinates, the solution is given by a single term, Eq. (8). The agreement shows that Eqs. (8) and (9) are the same solution only expressed in different forms.

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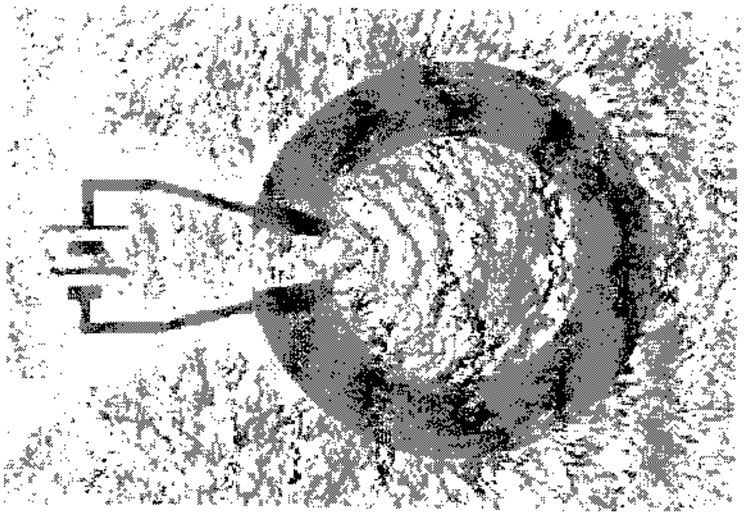

Figure 2: Lines of electric field mapped using grass seeds above painted glass plates, [4].

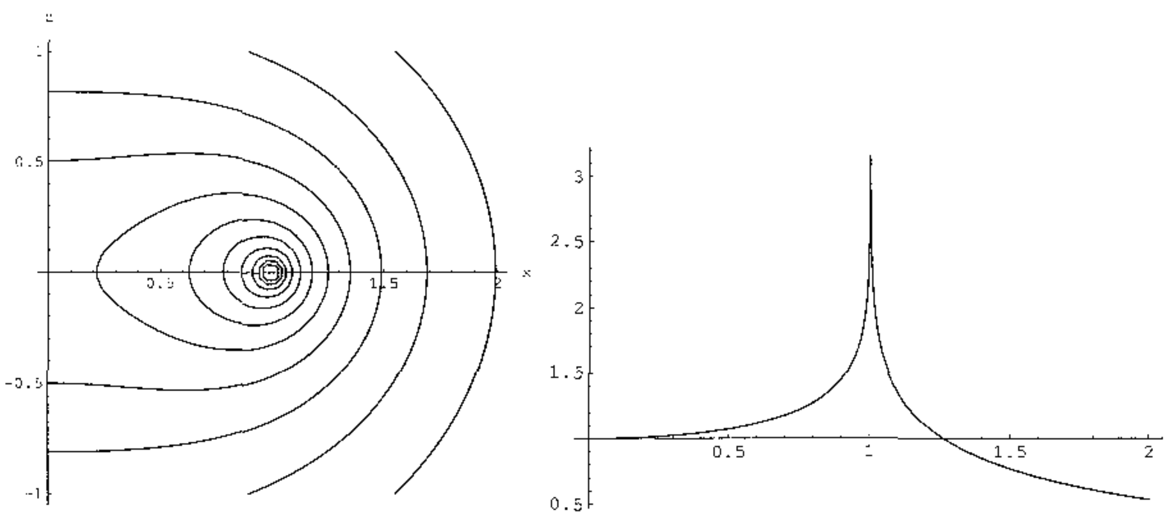

Figure 3: On the left are the equipotentials on the plane $y = 0$ (perpendicular to the ring) for the charged thin wire without current. Both Eq. (8) and (9) coincide. On the right, we have the electric potential as a function of the axial distance in the plane of the ring. We utilize $\eta_0 = 38$ ($\cosh \eta_0 = 1.6 \times 10^{16}$), $\phi_0 = 1/\phi(r = 0)$ and $a = 1$. Once more Eqs. (8) and (9) coincide.